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CONTROL ALLOCATION FOR OVERACTUATED SYSTEMS (PREPRINT)

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14. ABSTRACT

Much emphasis has been placed on overactuated systems for air vehicles. Overactuating an air vehicle provides a certain amount of redundancy for the flight control system, thus potentially allowing for recovery from off-nominal conditions. Due to this redundancy, control allocation algorithms are typically utilized to compute a unique solution to the overactuated problem. Control allocators compute the commands that are applied to the actuators so that a certain set of forces or moments are generated by the control effectors. Usually, control allocation problems are formulated as optimization problems so that all of the available degrees of freedom can be utilized and, when sufficient control power exists, secondary objectives can be achieved. In this work, a survey of control allocation techniques will be given.

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Control Allocation for Overactuated Systems

Michael W. Oppenheimer and David B. Doman

I. Abstract

Much emphasis has been placed on overactuated systems for air vehicles. Overactuating an air vehicle provides a certain amount of redundancy for the flight control system, thus potentially allowing for recovery from off-nominal conditions. Due to this redundancy, control allocation algorithms are typically utilized to compute a unique solution to the overactuated problem. Control allocators compute the commands that are applied to the actuators so that a certain set of forces or moments are generated by the control effectors. Usually, control allocation problems are formulated as optimization problems so that all of the available degrees of freedom can be utilized and, when sufficient control power exists, secondary objectives can be achieved. In this work, a survey of control allocation techniques will be given.

II. Introduction

Conventional aircraft utilize an elevator for pitch control, ailerons for roll control, and a rudder for yaw control. As aircraft design advances, more control effectors (some unconventional) are being placed on the vehicles. In some cases, certain control effectors may be able to exert significant influence upon multiple axes. When a system is equipped with more effectors than axes to control, the system may be overactuated or redundant. The allocation, blending, or mixing of these control effectors to achieve some desired objectives is the control allocation problem.

Due to overactuation and coupling of control surface effects, it is difficult to determine an appropriate method of how to translate a flight control command into a control surface command. In addition, rate and position limits of the control surfaces must be considered in order to achieve a realistic solution. Not only is the mixing of control surface effects critical, it is also desired to enable the aircraft to recover from off-nominal conditions, such as a failed control surface, when physically possible. In reconfigurable control systems, a control allocation algorithm is needed to perform automatic distribution of the control power requests among a large number of control effectors, while still obeying the rate and position limits of the actuators and to potentially allow recovery from off-nominal conditions.

Some of the simplest control allocation techniques are explicit ganging, pseudo control, pseudo inverse, and daisy chaining. Unfortunately, each suffers from difficulty in guaranteeing that rate and position limits will not be violated and some can be difficult to apply due to the need to derive a control mixing law a priori. Another control allocation method, called direct allocation [1], finds the control vector that results in the best approximation of the command vector in a given direction. Unconstrained least squares control allocation methods, that account for rate and position limits, through the use of penalty functions, have also been developed [2]. One of the first instances of linear programming based control allocators was from Paradiso [3]. In this work, Paradiso developed a selection procedure for determining actuator positions that was based on linear programming and limited actuator authority. More recently, the control allocation paradigm has been posed as a constrained optimization problem [4]. In this work, the control allocation problem was split into two sub-problems. The first was the error minimization part, which attempts to find the control vector, such that the control effector induced moments or accelerations match the desired moments or accelerations. If multiple solutions exist to the error minimization problem, the second problem attempts to find a unique solution by driving the control vector to a preference vector and optimizing a secondary objective. The linear control allocation problem has been extended to an affine problem [5] to account for nonlinearities in the moment-deflection curves. Quadratic programming has also been used in the past [6]. An excellent paper discussing control allocation, by Bodson [7], provides a glimpse into numerous control allocation techniques.

This paper will cover some of the more popular linear control allocation techniques. The objective is to present the methods in sufficient detail that the readers can utilize the techniques.

III. Linear Control Allocation

The linear control allocation problem can be posed as follows: find the control vector, $\boldsymbol{\delta} \in \mathbb{R}^n$, such that

$$\mathbf{B}\boldsymbol{\delta} = \mathbf{d}_{des} \tag{1}$$

subject to

$$\begin{aligned} \boldsymbol{\delta}_{min} \leq \boldsymbol{\delta} \leq \boldsymbol{\delta}_{max} \\ \dot{\boldsymbol{\delta}} \leq \dot{\boldsymbol{\delta}}_{max} \end{aligned} \tag{2}$$

where $\mathbf{B} \in \mathbb{R}^{m \times n}$ is a control effectiveness matrix, the lower and upper position limits are defined by $\boldsymbol{\delta}_{min} \in \mathbb{R}^n$ and $\boldsymbol{\delta}_{max} \in \mathbb{R}^n$, respectively, $\dot{\boldsymbol{\delta}} \in \mathbb{R}^n$ are the control rates, $\dot{\boldsymbol{\delta}}_{max} \in \mathbb{R}^n$ are the maximum control rates, $\mathbf{d}_{des}$ are the desired moments or accelerations (typically for inner-loop control laws, $\mathbf{d}_{des} \in \mathbb{R}^3$), n is the number of control

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effectors, and m is the number of axes to control. Equation 2 provides the position and rate limits for the control effectors. In a digital computer implementation, the rate limits are converted to effective position limits. The combined limits become the most restrictive of the rate or position limits and are specified as

$$\underline{\boldsymbol{\delta}} \leq \boldsymbol{\delta} \leq \overline{\boldsymbol{\delta}} \tag{3}$$

where

$$\begin{aligned} \overline{\boldsymbol{\delta}} &= \min\left(\boldsymbol{\delta}_{max}, \boldsymbol{\delta} + \Delta t \dot{\boldsymbol{\delta}}_{max}\right) \\ \underline{\boldsymbol{\delta}} &= \max\left(\boldsymbol{\delta}_{min}, \boldsymbol{\delta} - \Delta t \dot{\boldsymbol{\delta}}_{max}\right) \end{aligned} \tag{4}$$

Here, $\overline{\boldsymbol{\delta}} \in \mathbb{R}^n$, $\underline{\boldsymbol{\delta}} \in \mathbb{R}^n$, and $\overline{\boldsymbol{\delta}}$, $\underline{\boldsymbol{\delta}}$ are the most restrictive upper and lower control effector limits, respectively.

A necessary condition for a system to be overactuated is the number of columns of $\mathbf{B}$, n, must be greater than the number of rows of $\mathbf{B}$, m. The true test of overactuation is that the number of linearly independent columns of $\mathbf{B}$ be greater than the number of rows of $\mathbf{B}$. For inner-loop control laws, the $\mathbf{B}$, or control effectiveness, matrix typically becomes

$$\mathbf{B} = \begin{bmatrix} \frac{\partial L}{\partial \delta_1} \frac{\partial L}{\partial \delta_2} \cdots \frac{\partial L}{\partial \delta_n} \\ \frac{\partial M}{\partial \delta_1} \frac{\partial M}{\partial \delta_2} \cdots \frac{\partial M}{\partial \delta_n} \\ \frac{\partial N}{\partial \delta_1} \frac{\partial N}{\partial \delta_2} \cdots \frac{\partial N}{\partial \delta_n} \end{bmatrix} \tag{5}$$

where L, M, and N are the rolling, pitching, and yawing moments, respectively.

The control allocation problem can be illustrated with a simple example. Consider the following problem: find δ_1 and δ_2 such that

$$d_{des} = 3\delta_1 + \delta_2 \tag{6}$$

In this case, there is one objective function, $d_{des} \in \mathbb{R}^1$ and two controls. Using the form shown in Eq. 1 yields

$$\mathbf{B} = \begin{bmatrix} 3 & 1 \end{bmatrix} \quad \boldsymbol{\delta} = \begin{bmatrix} \delta_1 & \delta_2 \end{bmatrix}^T \tag{7}$$

Let the most restrictive limits be

$$-1 \leq \delta_1 \leq 1 \quad -1 \leq \delta_2 \leq 1 \tag{8}$$

Let $d_{des} = 2$. Then, this problem could be solved in a number of ways, for example, $\delta_1 = 0$ and $\delta_2 = 2$, or $\delta_1 = \frac{2}{3}$ and $\delta_2 = 0$, or $\delta_1 = -3$ and $\delta_2 = 11$, or many others. Graphically, this situation becomes as shown in Figure 1. The control allocation problem is to find δ_1, δ_2 such that $d_{des} = 3\delta_1 + \delta_2$ and the constraints are not violated. The solution is the intersection of the hyperspaces of the constraints and the equation $d_{des} = 3\delta_1 + \delta_2$. For $d_{des} = 2$ or 3, multiple solutions exist. For $d_{des} = 4$, only one solution exists, while for $d_{des} = 5$, no solutions exist. When only one solution exists, we simply select that solution. When multiple solutions exist, a method to pick one is necessary. When no solutions exist, a method to minimize the error, between d_{des} and $3\delta_1 + \delta_2$ is required, in some logical sense. This illustrates the control allocation problem.

Equations 1, 3, and 4 define the linear control allocation problem. The objective now is to determine methods which

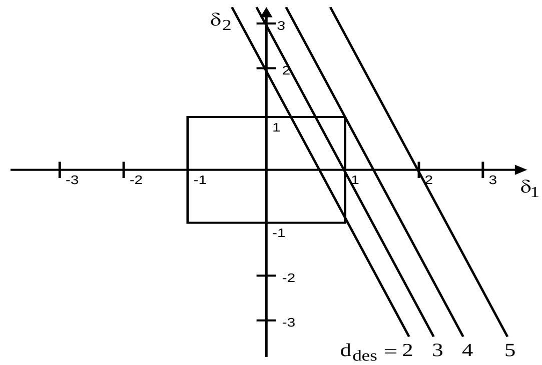

Fig. 1. Example of Control Allocation.

allow computation of the control effector vector $\boldsymbol{\delta}$, while possibly taking into account effector rate and position limits. The following discussion looks at different methods, which can be utilized to either reduce the dimension of the overactuated system to the point that a square allocation problem results or to directly solve the overactuated linear control allocation problem.

A. Explicit Ganging

In this approach, an a priori method is used to combine or gang the effectors to produce a single effective control from multiple devices. Note that historically, ganging was done with cables, pulleys, or other mechanical means. On modern aircraft, fly-by-wire is used and the ganging is performed in software. The goal is to find a matrix $\mathbf{G}$, that relates the pseudo controls, $\boldsymbol{\delta}_{pseudo}$, to the actual controls, $\boldsymbol{\delta}$ such that

$$\boldsymbol{\delta} = \mathbf{G}\boldsymbol{\delta}_{pseudo} \tag{9}$$

The vector $\boldsymbol{\delta}_{pseudo}$ are the pseudo controls, so named because some or all components of this vector are not physical control surfaces on the vehicle. For example, consider a vehicle that has left and right ailerons for roll control, $\delta_{a_L} \in \mathbb{R}^1$ and $\delta_{a_R} \in \mathbb{R}^1$, a single rudder for yaw control, $\delta_r \in \mathbb{R}^1$, and a single elevator for pitch control, $\delta_e \in \mathbb{R}^1$. A priori, a ganging law is constructed to produce a single roll control device. One possibility is to let

$$\delta_a = 0.5\left(\delta_{a_L} + \delta_{a_R}\right) \tag{10}$$

where $\delta_a \in \mathbb{R}^1$ is the single roll control device. Therefore, the full ganging law becomes

$$\begin{bmatrix} \delta_{a_L} \\ \delta_{a_R} \\ \delta_e \\ \delta_r \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \delta_a \\ \delta_e \\ \delta_r \end{bmatrix} \tag{11}$$

or

$$\boldsymbol{\delta} = \mathbf{G}\boldsymbol{\delta}_{pseudo} \tag{12}$$

Here, the reason for the term pseudo controls becomes clear because one of the elements of $\boldsymbol{\delta}_{pseudo}$, namely, δ_a, is not a physical control, instead, it is a linear combination of two physical control effectors, δ_{a_L} and δ_{a_R}. Then, the control allocation problem becomes find $\boldsymbol{\delta}_{pseudo}$, such that

$$\mathbf{B}\boldsymbol{\delta} = \mathbf{d}_{des} \Rightarrow \mathbf{B}\mathbf{G}\boldsymbol{\delta}_{pseudo} = \mathbf{d}_{des} \tag{13}$$

Solving this allocation problem for $\boldsymbol{\delta}_{pseudo}$ and using Eq. 12 yields the physical control effector commands. Typically, explicit ganging is used when it is obvious how to combine the redundant control effectors.

It is important to point out that this method can be used to reduce the control space dimension of an overactuated system. As previously mentioned, the inner-loop aircraft control law typically contains three objective functions, namely that the moments produced by the controls are equal to a set of desired moments ($\mathbf{d}_{des}$). In the example problem, $\boldsymbol{\delta} \in \mathbb{R}^4$, so that $\mathbf{B}\boldsymbol{\delta} = \mathbf{d}_{des}$ is a nonsquare allocation problem. After employing the explicit ganging methodology, $\boldsymbol{\delta}_{pseudo} \in \mathbb{R}^3$ so that $\mathbf{B}\mathbf{G}\boldsymbol{\delta}_{pseudo} = \mathbf{d}_{des}$ is a square allocation problem. Hence, the dimension of the control space is reduced from 4 to 3.

B. Pseudo Inverse

The pseudo inverse method is a constrained optimization technique that requires a pseudo inversion of the generally nonsquare $\mathbf{B}$ matrix. The pseudo inverse solution is the two-norm solution to the control allocation problem and can be formulated as follows:

$$\min_{\boldsymbol{\delta}} J = \min_{\boldsymbol{\delta}} \frac{1}{2}(\boldsymbol{\delta} + \mathbf{c})^{\mathbf{T}}\mathbf{W}(\boldsymbol{\delta} + \mathbf{c}) \tag{14}$$

subject to

$$\mathbf{B}\boldsymbol{\delta} = \mathbf{d}_{des} \tag{15}$$

where $\mathbf{W} \in \mathbb{R}^{n \times n}$ is a weighting matrix and $\mathbf{c} \in \mathbb{R}^n$ is an offset vector used to represent an off-nominal condition with one or more control effectors. To solve this problem, first find the Hamiltonian (H) such that

$$\begin{aligned} H = \frac{1}{2}\left(\boldsymbol{\delta}^T\mathbf{W}\boldsymbol{\delta} + \mathbf{c}^T\mathbf{W}\boldsymbol{\delta} + \boldsymbol{\delta}^T\mathbf{W}\mathbf{c} + \mathbf{c}^T\mathbf{W}\mathbf{c}\right) \\ + \boldsymbol{\xi}(\mathbf{B}\boldsymbol{\delta} - \mathbf{d}_{des}) \end{aligned} \tag{16}$$

where $\boldsymbol{\xi} \in \mathbb{R}^n$ is an as yet undetermined Lagrange multiplier. Taking the partial derivatives of H with respect to $\boldsymbol{\delta}$ and $\boldsymbol{\xi}$, setting these expressions equal to zero, and rearranging, gives

$$\begin{aligned} \frac{\partial H}{\partial \boldsymbol{\delta}} = \mathbf{W}\boldsymbol{\delta} + \frac{1}{2}(\mathbf{c}^{\mathbf{T}}\mathbf{W})^T + \frac{1}{2}\mathbf{W}\mathbf{c} + (\boldsymbol{\xi}\mathbf{B})^T = \mathbf{0} \\ \Longrightarrow \mathbf{W}\boldsymbol{\delta} = -\mathbf{W}\mathbf{c} - \mathbf{B}^{\mathbf{T}}\boldsymbol{\xi}^{\mathbf{T}} \end{aligned} \tag{17}$$

and

$$\begin{aligned} \frac{\partial H}{\partial \boldsymbol{\xi}} = \mathbf{B}\boldsymbol{\delta} - \mathbf{d}_{des} = \mathbf{0} \\ \Longrightarrow \mathbf{B}\boldsymbol{\delta} = \mathbf{d}_{des} \Longrightarrow \mathbf{B}\mathbf{W}^{-1}\mathbf{W}\boldsymbol{\delta} = \mathbf{d}_{des} \end{aligned} \tag{18}$$

Substituting Equation 17 into Equation 18 yields

$$\mathbf{B}\mathbf{W}^{-1}[-\mathbf{W}\mathbf{c} - \mathbf{B}^T\boldsymbol{\xi}^T] = \mathbf{d}_{des} \tag{19}$$

Solving for $\boldsymbol{\xi}^T$ in Equation 19 yields

$$\boldsymbol{\xi}^T = -(\mathbf{B}\mathbf{W}^{-1}\mathbf{B}^T)^{-1}[\mathbf{d}_{des} + \mathbf{B}\mathbf{c}] \tag{20}$$

Substituting Equation 20 into Equation 17 produces

$$\mathbf{W}\boldsymbol{\delta} = -\mathbf{W}\mathbf{c} + \mathbf{B}^T(\mathbf{B}\mathbf{W}^{-1}\mathbf{B}^T)^{-1}[\mathbf{d}_{des} + \mathbf{B}\mathbf{c}] \tag{21}$$

Simplifying Equation 21 gives the desired result

$$\begin{aligned} \boldsymbol{\delta} &= -\mathbf{c} + \mathbf{W}^{-1}\mathbf{B}^T(\mathbf{B}\mathbf{W}^{-1}\mathbf{B}^T)^{-1}[\mathbf{d}_{des} + \mathbf{B}\mathbf{c}] \\ &= -\mathbf{c} + \mathbf{B}^{\sharp}[\mathbf{d}_{des} + \mathbf{B}\mathbf{c}] \end{aligned} \tag{22}$$

where $\mathbf{B}^{\sharp}$ is the pseudo inverse of $\mathbf{B}$ and the superscript ($^{\sharp}$) indicates a pseudo inverse operation. Equation 22 gives the pseudo inverse solution. It should be noted that if an effector is offset, two items must be taken into account, position offset ($-\mathbf{c}$) and the moments generated by the offset ($\mathbf{B}\mathbf{c}$). For the position offset, the negative of the locked position is placed in the corresponding entry of the $\mathbf{c}$ vector. For example, assume there are four controls and control number 3 is stuck at $+5$ degrees. Then, the $\mathbf{c}$ vector would become

$$\mathbf{c} = \begin{bmatrix} 0 & 0 & -5 & 0 \end{bmatrix}^T \tag{23}$$

The weighting matrix, $\mathbf{W}$, can be selected to incorporate the position limits of the control effectors. For example, a diagonal element of $\mathbf{W}$ can be selected to be a function of the corresponding component of $\boldsymbol{\delta}$, so that the weighting function approaches ∞ as the control approaches a physical limit. There are no guarantees that commands to the control effectors will not exceed the position limits; however, in practice the method is effective in constraining the positions of the controls. As a final note, this method can be useful in generating preference vectors for more complex optimization based methods for the purpose of robustness analysis.

C. Redistributed Pseudo Inverse

The redistributed pseudo inverse works in a fashion similar to the pseudo inverse. The difference is that now the process is iterative and position saturated control effectors are removed from subsequent pseudo inverse solutions. The first step is to solve the control allocation problem using the pseudo inverse solution in Eq. 22 with $\mathbf{c}$ initially a vector of all zeros. If no controls exceed their minimum or maximum position limits, then the process stops and the solution from Eq. 22 is used. However, if one control saturates, the problem is solved again, this time zeroing out the column of the $\mathbf{B}$ matrix corresponding to the saturated control and placing the negative of the saturated value in the vector $\mathbf{c}$. One must be careful here. When saturation occurs, there are two $\mathbf{B}$ matrices in Eq. 22, one for the pseudo inverse solution $\mathbf{W}^{-1}\mathbf{B}^T(\mathbf{B}\mathbf{W}^{-1}\mathbf{B}^T)^{-1}$ and one for the offset or saturated contribution $\mathbf{B}\mathbf{c}$. When zeroing out a column corresponding to a saturated effector, only the pseudo inverse $\mathbf{B}$ matrix is modified, while the $\mathbf{B}\mathbf{c}$ term uses the original $\mathbf{B}$ matrix. Consider the following example of a redistributed pseudo inverse control allocation problem. Let

$$\mathbf{B} = \begin{bmatrix} 2 & -2 & -2 & -1 \\ 1 & 1 & -3 & -2 \\ 2 & -2 & -1 & -1 \end{bmatrix} \quad \mathbf{d}_{des} = \begin{bmatrix} .5 & 1 & 1 \end{bmatrix}^T \tag{24}$$

$$-0.75 \leq \delta_{1,2,3,4} \leq 0.75 \tag{25}$$

In this example, there are 3 objective functions (3 rows in $\mathbf{B}$), four control effectors (4 columns in $\mathbf{B}$), each control has the same lower and upper position limits, and the weighting matrix is $\mathbf{W} = \mathbf{I}$. The first step in the redistributed pseudo inverse solution is to compute Eq. 22. The results are

$$\boldsymbol{\delta} = \mathbf{B}^{\sharp}\mathbf{d}_{des} == \begin{bmatrix} 0.55 & 0.23 & 0.5 & -0.86 \end{bmatrix}^T \tag{26}$$

where the $\mathbf{c}$ vector was removed since it currently contributes nothing. Since δ_4 exceeds it limit, force δ_4 to -0.75 and zero out the fourth column of the $\mathbf{B}$ matrix. Now, calculate the new pseudo inverse solution using

$$\boldsymbol{\delta} = -\mathbf{c} + \mathbf{B}^{\sharp}_{red}\left[\mathbf{d}_{des} - \mathbf{Bc}\right] = \begin{bmatrix} .6875 & .3125 & .5 & -.75 \end{bmatrix}^T \tag{27}$$

where $\mathbf{B}_{red}$ is the reduced $\mathbf{B}$ matrix formed by zeroing out the column corresponding to the saturated control effector. The results show that δ_4 is now at its most negative limit, as expected, and no other control effectors exceed their limit. The iterations stop and the result of the redistributed pseudo inverse calculation is

$$\boldsymbol{\delta} = \begin{bmatrix} .6875 & .3125 & .5 & -.75 \end{bmatrix}^T \tag{28}$$

A check of this result gives

$$\mathbf{B}\begin{bmatrix} .6875 & .3125 & .5 & -.75 \end{bmatrix}^T = \begin{bmatrix} 0.5 & 1 & 1 \end{bmatrix}^T = \mathbf{d}_{des} \tag{29}$$

Thus, the calculated control settings do provide the desired commands. In this example, only 2 iterations of pseudo inverse calculations were performed. If the control effector setting computed in Eq. 27 had one or more controls exceeding their position limits, then the process of zeroing out columns in the $\mathbf{B}$ matrix would continue until all controls are at their limits (either positive or negative) or a feasible solution is found (one where remaining controls do not exceed position limits).

D. Daisy Chaining

The daisy chain assumes a hierarchy of control effectors. In this method, when one control or a group of controls saturates, there is an error between the commanded moments or accelerations and those produced by the control effectors. The daisy chain method would then utilize another control to produce the required moments or accelerations that are lacking due to the saturation of a control effector. Figure 2 shows an example of daisy chain allocation. In this example, the goal is to produce a desired pitch acceleration, given by $\dot{q}_{des}$. There are three controls that can produce pitching moment, an elevator (δ_e), a bodyflap (δ_{bf}), and a canard (δ_c). The daisy chain procedure works as follows: the primary control effector, δ_e in this case, is commanded to produce the desired acceleration. If the elevator can produce this acceleration, then nothing else happens and the bodyflap and canard are not utilized. However, if there is a moment deficiency between the acceleration that the elevator produces and the desired acceleration, the control effector, which is second in line, namely the bodyflap, is commanded to produce an acceleration equivalent to the acceleration deficiency. If the bodyflap can produce the required acceleration, then the canard does nothing. However, if the bodyflap cannot produce the required acceleration, then the canard is commanded to produce the difference between the commanded acceleration and the accelerations produced by the elevator and bodyflap. This method is valid for any number of control effectors.

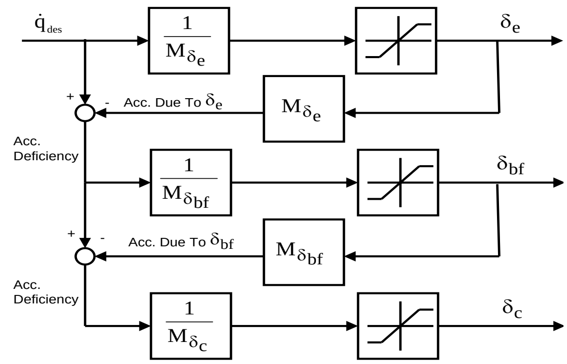

Fig. 2. Example of Daisy Chain Allocation.

E. Direct Allocation

The direct allocation method, by Durham [1], is a constrained control allocation method aimed at finding a real number, ρ, and a vector, $\boldsymbol{\delta}_1$, such that

$$\mathbf{B}\boldsymbol{\delta}_1 = \rho\mathbf{d}_{des} \tag{30}$$

and

$$\underline{\boldsymbol{\delta}} \leq \boldsymbol{\delta} \leq \overline{\boldsymbol{\delta}} \tag{31}$$

If $\rho > 1$, let $\boldsymbol{\delta} = \frac{1}{\rho}\boldsymbol{\delta}_1$. If $\rho \leq 1$, let $\boldsymbol{\delta} = \boldsymbol{\delta}_1$. In order to use this method, an attainable moment set [9] (AMS) must be established. The AMS is of the dimension of $\mathbf{d}_{des}$ and for linear systems of the form of Eq. 30, consists of a convex hull with planar surfaces. Physically, ρ represents how much a control power demand must be scaled to touch the boundary of the AMS. When $\rho \leq 1$, the moment demand lies within the AMS and the allocator supplies the demand. When $\rho > 1$, the control power demand exceeds supply and the demand is scaled back to touch the boundary of the AMS. Algorithms exist for generating the AMS for the 3 moment problem [9] and the direct allocation problem extends to nonlinear problems of the form $f(\boldsymbol{\delta}) = \rho\mathbf{d}_{des}$. Unfortunately, the construction of the AMS for a nonlinear problem can be extremely difficult.

F. Error and Control Minimization

The objective of the error minimization, control law command feasibility, or control deficiency problem is to find a vector $\boldsymbol{\delta}$, given a matrix $\mathbf{B}$, such that

$$\min_{\boldsymbol{\delta}} J = \|\mathbf{B}\boldsymbol{\delta} - \mathbf{d}_{des}\| \tag{32}$$

is minimized, subject to

$$\underline{\boldsymbol{\delta}} \leq \boldsymbol{\delta} \leq \overline{\boldsymbol{\delta}} \tag{33}$$

The norm used would depend on the type of algorithm used to perform the minimization. When used with linear programming solvers, the error minimization problem is specified as

$$\min_{\boldsymbol{\delta}} J = \|\mathbf{B}\boldsymbol{\delta} - \mathbf{d}_{des}\|_1 \tag{34}$$

subject to the constraints specified in Eq. 33. This can be transformed into the standard linear programming problem [4]

$$\min_{\boldsymbol{\delta}} J = \left[\begin{array}{cccccc} 0 & \cdots & 0 & 1 \cdots & 1 \end{array} \right] \left[\begin{array}{c} \boldsymbol{\delta} \\ \boldsymbol{\delta}_s \end{array} \right] \tag{35}$$

subject to

$$\left[\begin{array}{c} \boldsymbol{\delta}_s \\ -\boldsymbol{\delta} \\ \boldsymbol{\delta} \\ -\mathbf{B}\boldsymbol{\delta} + \boldsymbol{\delta}_s \\ \mathbf{B}\boldsymbol{\delta} + \boldsymbol{\delta}_s \end{array} \right] \geq \left[\begin{array}{c} \mathbf{0} \\ -\overline{\boldsymbol{\delta}} \\ \underline{\boldsymbol{\delta}} \\ -\mathbf{d}_{des} \\ \mathbf{d}_{des} \end{array} \right] \tag{36}$$

where $\boldsymbol{\delta}_s \in \mathbb{R}^m$ is a vector of slack variables. Note that the slack variables must be positive, but are otherwise unconstrained. Individually, the slack variables represent how much control power demand exceeds supply in any given axis. If $J = 0$, then the control law command is feasible, otherwise it is infeasible. The solution to Eqs. 35 and 36 can be obtained with a linear programming solver.

The control minimization or control sufficiency problem is a secondary optimization. If there exists sufficient control authority to satisfy Eq. 34, that is, if $J = 0$, then a secondary objective may be achieved. The ability to do this is a direct result of the overactuated system, in that multiple solutions to the problem may exist and one solution may be preferred over another. The control minimization is posed as follows:

$$\min_{\boldsymbol{\delta}} J = \mathbf{W}_u^T \boldsymbol{\delta}_s \tag{37}$$

subject to

$$\left[\begin{array}{c} \boldsymbol{\delta}_s \\ -\boldsymbol{\delta} \\ \boldsymbol{\delta} \\ -\boldsymbol{\delta} + \boldsymbol{\delta}_s \\ \boldsymbol{\delta} + \boldsymbol{\delta}_s \end{array} \right] \geq \left[\begin{array}{c} \mathbf{0} \\ -\overline{\boldsymbol{\delta}} \\ \underline{\boldsymbol{\delta}} \\ -\boldsymbol{\delta}_p \\ \boldsymbol{\delta}_p \end{array} \right] \tag{38}$$

$$\mathbf{B}\boldsymbol{\delta} = \mathbf{d}_{des}$$

where $\mathbf{W}_u^T \in \mathbb{R}^{m \times 1}$, $\boldsymbol{\delta}_s \in \mathbb{R}^m$, and $\boldsymbol{\delta}_p \in \mathbb{R}^n$ is the preferred control effector position vector. So, the first requirement is that the moment or acceleration demand is met, followed by selecting the control effector positions which satisfy the moment demand and minimize a secondary objective. Many secondary objectives exist, for example, minimum control deflection, minimum wing loading, minimum radar signature, minimum drag, minimum actuator power, and others. A few of these secondary objectives are discussed in [4].

G. Minimum 2-Norm for Robustness Analysis

One of the drawbacks of the optimization based linear programming methods is that it is impossible to represent this allocator in a useful format. The control allocation algorithms have $\mathbf{d}_{des}$ as an input and $\boldsymbol{\delta}$ as an output. In the pseudo inverse development, a direct relationship exists between inputs and outputs. When $\mathbf{c} = \mathbf{0}$ and $\mathbf{W} = \mathbf{I}$, Eq. 22 becomes

$$\boldsymbol{\delta} = \mathbf{B}^T(\mathbf{B}\mathbf{B}^T)^{-1}[\mathbf{d}_{des}] = \mathbf{B}^\sharp \mathbf{d}_{des} \tag{39}$$

In this case, the control allocator is simply a gain matrix given by $\mathbf{B}^\sharp$. This is useful for stability or robustness analysis because a model of the control allocator, namely $\mathbf{B}^\sharp$, has been determined. When linear programming control allocators are utilized, no such model exists and the relationship between inputs ($\mathbf{d}_{des}$) and outputs ($\boldsymbol{\delta}$) is more complicated. In this case, determining stability or robustness of closed loop systems is not feasible. Fortunately, there are situations where a linear programming control allocator can be modelled. If no control effectors exceed their rate and position limits, then driving the preference vector in Eq. 38 to the 2-norm (or pseudo inverse solution), given by Eq. 39, provides a method to model the allocator. This is because the solution to the control allocation problem will be the 2-norm solution and hence, the linear programming control allocator block can be replaced by $\mathbf{B}^\sharp$ for use in stability or robustness analysis. When control effectors exceed rate or position limits, this model is no longer valid. Note that this technique can also be used in the mixed optimization problem described in the next section.

H. Mixed Optimization

The mixed optimization problem is a combination of the control and error minimization problems. The mixed optimization problem is posed as follows: given a control effectiveness matrix $\mathbf{B}$ and a preferred control vector $\boldsymbol{\delta}_p$, find a vector $\boldsymbol{\delta}$ such that

$$J = \|\mathbf{B}\boldsymbol{\delta} - \mathbf{d}_{des}\| + \nu\|\boldsymbol{\delta} - \boldsymbol{\delta}_p\| \tag{40}$$

is minimized, subject to $\underline{\boldsymbol{\delta}} \leq \boldsymbol{\delta} \leq \overline{\boldsymbol{\delta}}$ where $\nu \in \mathbb{R}^n$ is a factor which weights the relative importance of the error and control minimization problems. Bodson [7] has converted the mixed optimization problem into a standard linear program.

I. Affine Control Allocation

A linear control allocation problem (as described in Eq. 1), suffers from the fact that it assumes that the individual entries in the control effectiveness (slopes of moment-deflection data) pass through the origin. In a local sense, this is typically not the case and it is directly attributable to nonlinear effects in the moment-deflection relationship. A more accurate solution to the control allocation problem is obtained using an affine control allocation problem formulation [5], that is, one of the form: find $\boldsymbol{\delta}$ such that

$$\mathbf{B}\boldsymbol{\delta} + \boldsymbol{\epsilon} = \mathbf{d}_{des} \tag{41}$$

is minimized, subject to

$$\overline{\boldsymbol{\delta}} \leq \boldsymbol{\delta} \leq \underline{\boldsymbol{\delta}} \tag{42}$$

where $\boldsymbol{\epsilon}$ is an intercept term that provides a more robust control allocation algorithm when the moment-deflection curves are not truly linear.

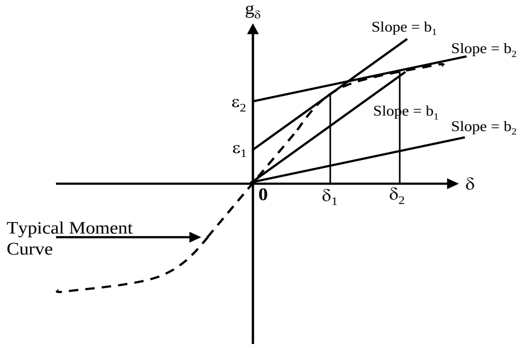

Fig. 3. Typical Moment Curve and Slopes at Operating Conditions.

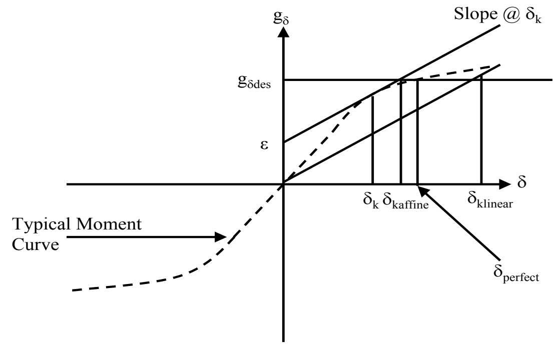

Fig. 4. Linear Versus Affine Control Effectiveness.

To begin the analysis of the intercept correction term, consider a representative one-dimensional moment deflection curve shown in Fig. 3. The horizontal axis is the control effector position, δ, while the vertical axis is the moment produced by the control effector, g_δ. At the operating points, δ_1 and δ_2, the control effectiveness is given by the slopes b_1 and b_2, while the intercepts of the moment axis are ϵ_1 and ϵ_2. In the linear control allocation problem, the moment axis intercepts are zero, as shown by the slope b_1 and b_2 control effectiveness lines that pass through the origin. When the control is operating at the position given by δ_k (see Fig. 4), the effectiveness is given by the tangent to the moment-deflection curve at that operating condition. In a purely linear control allocation problem, this slope is translated along the vertical axis until it intersects the origin and, as shown in Fig. 4, can have adverse effects on the computation of δ required to provide the desired moment. Here, we begin by assuming that the control is currently operating at δ_k and draw the tangent to the moment-deflection curve at that point and also translate this tangent line to the origin. When the next moment command is provided, assume that the command is $g_{\delta_{des}}$. If the control allocation algorithm had perfect knowledge of the moment-deflection curve, then the algorithm could accurately compute $\delta_{perfect}$ as the required control position to produce the desired moment. However, in a linear (or affine) allocation problem, only moment-deflection slopes are available to the allocation algorithm. If the linear case is utilized, then a large error exists between $\delta_{perfect}$ and the position computed using linear information, δ_{linear}. On the other hand, if an affine representation of the moment-deflection curve is utilized, the error between $\delta_{perfect}$ and δ_{affine} is much less than the error between $\delta_{perfect}$ and δ_{linear}. Hence, this method can produce significantly more accurate results for the computation of δ.

It should be noted that the affine method is well suited to algorithms which obey the rate limits of the effectors and are implemented in digital computers. The rate limits and digital implementation essentially limit the distance a control effector can travel in one timestep. Therefore, even if operating in a nonlinear region of the moment-deflection curve, the rate limit restrictions allow the control effectiveness (tangent of moment-deflection curve) to be an accurate representation of the nonlinear curve.

IV. Conclusions

This work provided an overview of some of the techniques used to address the linear control allocation problem. Methods to reduce the dimension of the control effector space, such as, explicit ganging and daisy chaining, were presented. More sophisticated algorithms, that take into account control effector rate and position limits, were also discussed. In these cases, linear programming has proved to be a viable method for solving online control allocation problems.

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